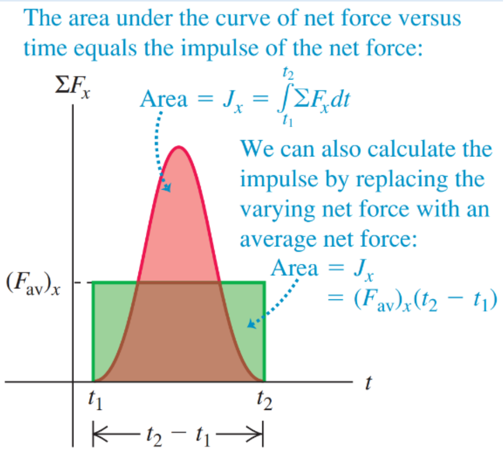
Impulse and theorem of momentum 冲量和动量定理:



Momentum 动量:

$$\vec{P} = m\vec{v}$$

Impulse 冲量:

$$\vec{J} = \sum \vec{F}\,\Delta t = \sum \vec{F}\,(t_2 - t_1)$$

$$\vec{J} = \int \sum \vec{F}\,dt$$

Impulse-Momentum Theorem

$$\vec{J} = \vec{P}_2 - \vec{P}_1 \quad \text{or}$$

$$\int_{t1}^{t2} \sum \vec{F}\,dt = \int_{\vec{P}_1}^{\vec{P}_2} d\vec{P} = \vec{P}_2 - \vec{P}_1$$

Average force $\vec{F}_{av}$: $\quad \vec{J} = \vec{F}_{av}(t_2 - t_1)$

Example: A $2kg$ object is throwing toward a wall. During the collision with the wall lasting from $t = 0$ to $t = 0.2s$, the force acting on the object is given by the equation $\vec{F} = 300t(0.2 - t)\vec{\imath}$ (N)

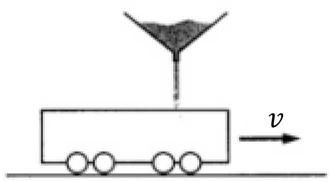
(a) The impulse that the force acts on the object during the collision is:

$$\vec{J} = \int \sum \vec{F}\,dt = \int_0^{0.2} 300t(0.2 - t)\vec{\imath}\,dt = (30t^2 - 100t^3)_0^{0.2} = 0.4 \ (kgm/s)$$

(b) The average force on the object is:

$$\vec{F}_{av} = \frac{\vec{J}}{t_2 - t_1} = \frac{0.4}{0.2 - 0} = 2 \ (N)$$

Example: An open-top railroad car (initially empty and of mass m_0) rolls with negligible friction along a straight horizontal track and passes under the spout of a sand conveyor. When the car is under the conveyor, sand is dispensed from the conveyor in a narrow stream at a steady rate $dm/dt = C$ and falls vertically to the car. The car has initial speed v_0 and sand is filling it from time $t = 0$ to $t = T$.



(a) Calculate the mass m of the car plus the sand that it catches as a function of time t from 0<t<T.

Weight of sand: $dm = Cdt \Rightarrow \int_0^{m_s} dm = \int_0^T Cdt \Rightarrow m_s = Ct$

Total weight $m = m_s + m_0 = Ct + m_0$

(b) Determine the speed v of the car as a function of time t from 0<t<T.

In the horizontal direction, as the friction is negligible, the Momentum of the car-sand system keeps constant:

$$P = m_0 v_0 = (Ct + m_0)v(t) \Rightarrow \mathrm{v(t)} = \frac{m_0 v_0}{m_0 + Ct}$$

(c) If you want to keep the velocity of the car constant, what kind of force needs to act on the car.

To keep the sand-car system's velocity constant, an impulse must be applied to the sand:

$$\int_0^t F dt = \int_0^m v_0 dm = \int_0^t v_0 C dt \Rightarrow Ft = v_0 Ct \Rightarrow F = Cv_0$$

Exercise 11: A bullet shoot out of a gun at $v_0\ (m/s)$. While the bullet accelerates in the barrel of the gun, the total force that is applied on it is

$$F = a - bt \qquad (a, b \text{ are constant and } t \text{ is in second})$$

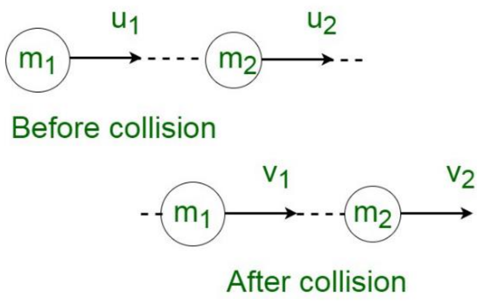
(a) Suppose the total force that the bullet is subjected to is zero upon the exit of the gun, calculate the total time that the bullet takes to run through the barrel.

(b) Calculate the momentum of the bullet.

(c) Calculate the mass of the bullet.

Collision in one dimension(Central impact) 一维碰撞

General Case:



$$m_1u_1 + m_2u_2 = m_1v_1 + m_1v_2$$

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

$$\Rightarrow \begin{cases} v_1 = \dfrac{m_1 - em_2}{m_1 + m_2}u_1 + \dfrac{(1+e)m_2}{m_1 + m_2}u_2 \\ v_2 = \dfrac{(1+e)m_1}{m_1 + m_2}u_1 - \dfrac{em_1 - m_2}{m_1 + m_2}u_2 \end{cases}$$

此公式不用背

Kinetic energy lost during the collision:

$$\Delta E_{k-lost} = E_{ki} - E_{kf} = \frac{1}{2}(1-e^2)\frac{m_1m_2}{m_1+m_2}(u_1 - u_1)^2$$

$e = 0$: Perfectly inelastic collision. ΔE_{k-lost} get the maximum value.

$e = 1$: Elastic collision. $\Delta E_{k-lost} = 0$

$0 < e < 1$: Inelastic collision.

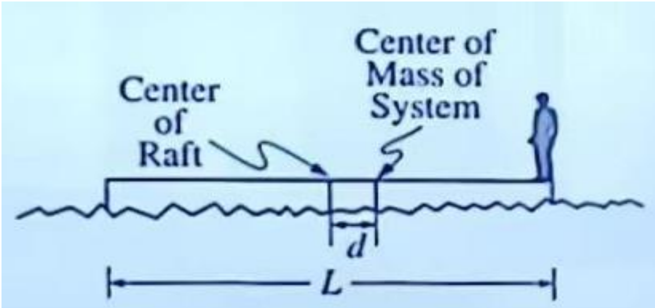
Center of mass 质心

Center of mass 质心:

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

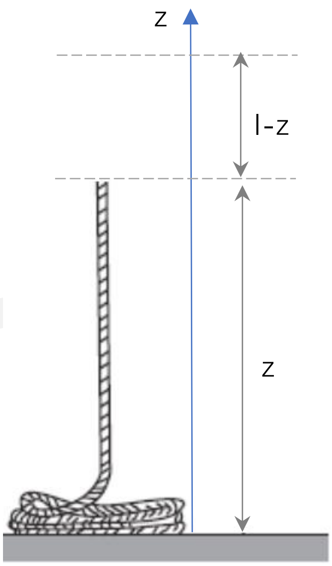
Example 1: A person is standing at one end of a uniform raft of length L that is floating motionless on water, as shown. The center of mass of the person-raft system is a distance d from the center of the raft. The person then walks to the other end of the raft. If friction between the raft and the water is negligible, how far does the draft move relative to the water?



$$x = 2d$$

Center of mass reference frame 质心参考系

A uniform soft rope with mass m and length l hangs vertically with its lower end just touching the ground. The rope is released from rest. Try to calculate the force of the ground acting on the rope when the remaining length in the air is z.



Momentum s 动量: $\vec{p}_{sys} = m\vec{v}_{cm}$

牛顿第二定律: $\sum \vec{F}_{ext} = m\vec{a}_{cm}$

$$\lambda = \frac{m}{l}$$

Center of mass location

$$z_{cm} = \frac{1}{m}\int_0^z \lambda z\,dz = \frac{1}{m}\int_0^z \frac{m}{l} z\,dz = \frac{z^2}{2l}$$

$$v_{cm} = \frac{dz_{cm}}{dt} = \frac{z}{l}\left(\frac{dz}{dt}\right)$$

$$a_{cm} = \frac{dv_{cm}}{dt} = \frac{1}{l}\left(\frac{dz}{dt}\right)^2 + \frac{z}{l}\left(\frac{d^2z}{dt^2}\right) = \frac{2g(l-z)}{l} - \frac{gz}{l} = 2g - 3g\frac{z}{l}$$

$$\frac{dz}{dt} = v_z = -\sqrt{2g(l-z)}$$

$$\frac{d^2z}{dt^2} = a_z = -g$$

此方法的关键点是写出v_z和a_z的表达式。代入上式

这种方法有点难。

$$ma_{cm} = F - mg \Rightarrow F = ma_{cm} + mg = m\left(2g - 3g\frac{z}{l}\right) + mg = 3mg\frac{l-z}{l}$$

Newton's 2nd law with continuously variable mass 连续变化质量的牛顿第二定律

$$\vec{F}_{ext} + \frac{\mathrm{dm}}{\mathrm{dt}}(\vec{u} - \vec{v}) = m\frac{d\vec{v}}{dt}$$

dm: 在时间 dt 内系统变化的质量(增加为+减少为-)

$\vec{u} - \vec{v}$ 此变化质量的末速度-初速度(注意方向和符号)

Physics for Scientists and Engineers 6ed (Paul A. Tipler, Gene Mosca), p273

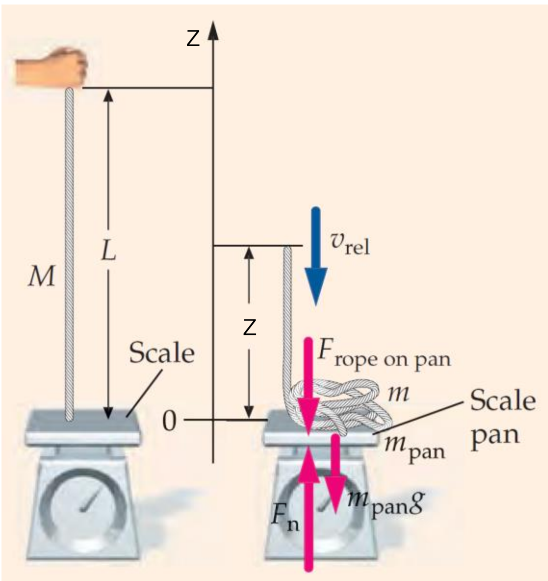
研究系统 { 已经落在称上的绳子
时间dt内新落下的绳子 }

解题思路:

1. 已经落下的绳子←落下长度是 l-z, 质量是 m(l-z)/l
2. 时间 dt 内落下的绳子 dm

   ←dm 可用 dt 时间内落下的长度 dz 计算出

   ←dz 可以用 dt 开始时刻 dm 的速度 v 和 dt 表示出



←dm 下降了(l-z),← $\frac{1}{2}(dm)v^2 = dm(g)(l-z)$

Solution

1. For the Newton's 2nd law with continuously variable mass

$$F_{ext} + \frac{\mathrm{dm}}{\mathrm{dt}}[u - (-v)] = \left[m\left(\frac{l-z}{l}\right)\right]\frac{dv}{dt}$$

2. $u = 0,\ \frac{1}{2}(dm)v^2 = dm(g)(l-z) \Rightarrow v = -\sqrt{2g(l-z)}$

3. $dz = vdt, dm = dz\left(\frac{m}{l}\right) = vdt\left(\frac{m}{l}\right)$

4. $\frac{dv}{dt} = -g$:在落地前作自由落体运动

$$F_{ext} = -\frac{\mathrm{dm}}{\mathrm{dt}}v + \left[m\left(\frac{l-z}{l}\right)\right]\frac{dv}{dt} = -v\left(\frac{m}{l}\right)v - \left[m\left(\frac{l-z}{l}\right)\right]g$$

$$= -v^2\left(\frac{m}{l}\right) - mg\left(\frac{l-z}{l}\right) = -[2g(l-z)] - mg\left(\frac{l-z}{l}\right) = -3mg\frac{l-z}{l}$$

Key Point:

1. 选取的研究对象是已经落在秤上的绳子和时间 dt 内落下绳子的总和
2. dm/dt, (u-v), dv/dt 这三个量的正负号很容易出错。

目录